Lecture 10

Terrestrial infrared radiative processes. Part 3:

Absorption band models. Curtis-Godson Approximation.

Objectives:

1. Concept of the equivalent width. Limits of the strong and weak lines.

2. Absorption-band models: Regular (Elsasser) band model and Statistical (Goody) band model.

3. Curtis-Godson Approximation for inhomogeneous path.

Required reading:

L02: 4.4

Additional/Advanced reading:

G&Y: 4.5;4.6

1. Concept of the equivalent width. Limits of the strong and weak lines.

First, let's consider **a homogeneous atmospheric layer** (i.e., the spectral absorption coefficient $\boldsymbol{k_\nu}$ does not depend on path length).

Recall Lecture 8 where we have defined the **spectral transmission function** for a band of a width $\boldsymbol{\Delta\nu}$ as

$$T_{\bar{\nu}}(u) = \frac{1}{\Delta\nu}\int_{\Delta\nu} \exp(-k_\nu u)d\nu = \frac{1}{\Delta\nu}\int_{\Delta\nu} \exp(-Sf(\nu-\nu_0)u)d\nu$$

and **spectral absorptance**

$$A_{\bar{\nu}}(u) = 1 - T_{\bar{\nu}}(u) = \frac{1}{\Delta\nu}\int_{\Delta\nu}(1-\exp(-k_\nu u))d\nu$$

Equivalent width is defined as

$$\boxed{W(u) = A_{\bar{\nu}}\Delta\nu = \int_{\Delta\nu}[1-\exp(-k_\nu u)]d\nu} \qquad [10.1]$$

where W is in units of wavenumber (cm^{-1}).

- The equivalent width is the width of a fully absorbing (A=1) rectangular-shape line.

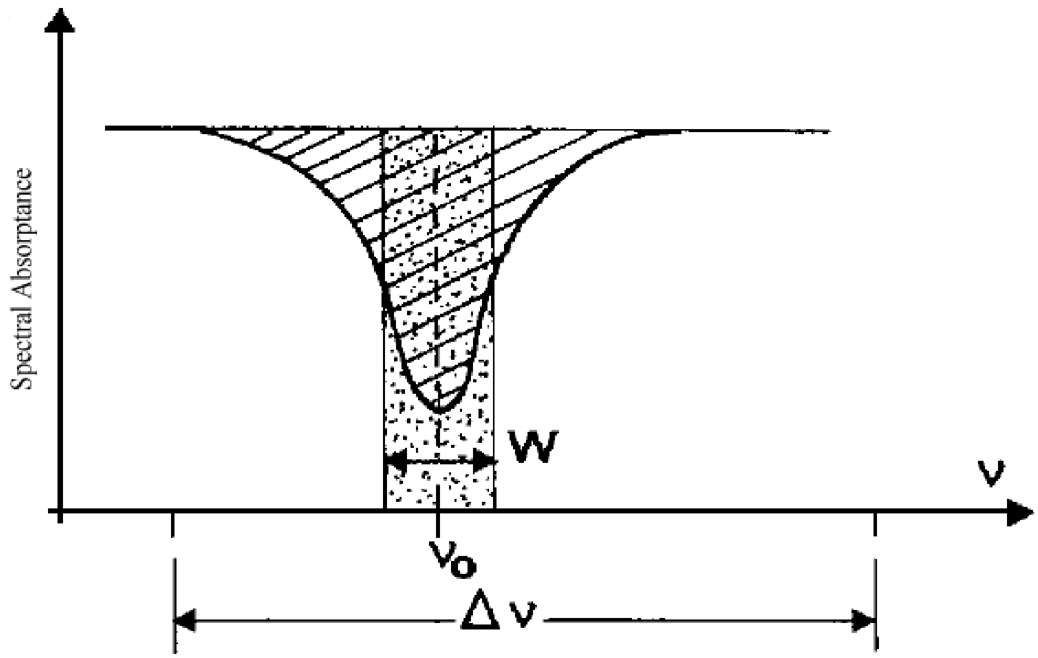

Figure 10.1 Schematic illustration of the equivalent width. The dotted rectangular area is equal to the hatched area and represents the total energy absorbed in the line.

- **<u>Equivalent width of Lorentz profile</u>**

Using $k_\nu = S\, f(\nu - \nu_0)$ and the Lorentz profile of a line, we have

$$A_{\bar{\nu}}(u) = \frac{1}{\Delta\nu}\int_{\Delta\nu}(1 - \exp\left(-\frac{S\alpha u/\pi}{(\nu-\nu_0)^2+\alpha^2}\right))d\nu \qquad [10.2]$$

This integral can be expressed in term of the Ladendurg and Reiche function, L(x), as

$$\boxed{W = A_{\bar{\nu}}\Delta\nu = 2\pi\alpha L(x)} \qquad [10.3]$$

where $\boldsymbol{x = Su/2\pi\alpha}$,

S is the line intensity, and **u** is the absorber amount.

NOTE: The Ladendurg and Reiche function L(x) in Eq.[10.3] is given by the modified Bessel functions of the first kind of order n: $L(x) = x\exp(-x)[I_0(x) + I_1(x)]$, where

$$I_n(x) = i^{-n} J_n(ix) \quad \text{and} \quad J_n(x) = \frac{i^{-n}}{\pi} \int_0^{\pi} \cos(n\theta)\ \exp(ix\ \cos(n\theta)) d\theta$$

For small x: L(x) is linear with its asymptotic expansion: L(x)= x[1-…]

For large x: L(x) is proportional to a square root of x: L(x) = $(2x/\pi)^{1/2}$[1-…]

Case of weak line absorption: either k_ν or u is small => $\boldsymbol{k_\nu u \ll 1}$

Using the asymptotic of L(x) for small x, we have

$$A_{\bar{\nu}}(u) = \frac{W}{\Delta\nu} = 2\pi\alpha L(x) / \Delta\nu = 2\pi\alpha \frac{Su}{2\pi\alpha\Delta\nu} = \frac{Su}{\Delta\nu}$$

Thus

$$\boxed{A_{\bar{\nu}}(u) = \frac{Su}{\Delta\nu}}$$ is called Linear absorption law. [10.4]

Case of strong line absorption: $\boldsymbol{Su/\pi\alpha \gg 1}$

Using the asymptotic of L(x) for large x, we have

$$A_{\bar{\nu}}(u) = \frac{W}{\Delta\nu} = 2\pi\alpha L(x) / \Delta\nu = 2\pi\alpha \sqrt{\frac{2x}{\pi}} / \Delta\nu =$$

$$= 2\pi\alpha \sqrt{\frac{2Su}{\pi 2\pi\alpha}} / \Delta\nu = 2\sqrt{Su\alpha} / \Delta\nu$$

Thus

$$\boxed{A_{\bar{\nu}}(u) = 2\frac{\sqrt{Su\alpha}}{\Delta\nu}}$$ is called Square root absorption law. [10.5]

2. Absorption band models.

Band is a spectral interval of a width **Δν** which is small enough to utilize a mean value of the Plank function $B_{\bar{\nu}}(T)$, but large enough so it consists of several **absorption lines**.

- **Absorption band models** are introduced to simplify the computation of the spectral transmittance. Some generally available radiative transfer codes (such as MODTRAN) use band models.

NOTE: MODTRAN is a moderate resolution radiative transfer code, which has "fixed-wavenumber" sampling of 1 cm^{-1} and a nominal resolution of 2 cm^{-1}. See A. Berk, G.P. Anderson, P.K. Acharya, J.H. Chetwynd, L.S. Bernstein, E.P. Shettle, M.W. Matthew, and S.M. Adler-Golden, MODTRAN4 USER'S MANUAL. Air Force Research Laboratory. 2000.

Let's consider a band with several lines. Two main cases can be identified:

1) lines have **regular positions**

2) lines have **random positions**.

Two main types of band models: **regular** band model and **random** band models.

Regular Elsasser band model consists of an infinite array of Lorentz lines of equal intensity, spaced at equal intervals.

Example: This type of bands is similar to P and Q branches of linear molecules. For example, the spectrum of N_2O in 7.78 μm band; the spectrum of CO_2 in 15 μm band.

The **absorption coefficient** of the Elsasser bands is

$$k_\nu = \sum_{n=-\infty}^{\infty} \frac{S}{\pi} \frac{\alpha}{(\nu - n\delta)^2 + \alpha^2} \qquad [10.6]$$

where δ is the line spacing (i.e., the distance in wavenumber domain (cm^{-1}) between the centers of two nearest lines).

Using Eq.[10.6} one can calculate the spectral absorptance as (see derivation in L02 pp139-141)

$$A_{\bar{\nu}} = erf\left(\frac{\sqrt{\pi S \alpha u}}{\delta}\right) \qquad [10.7]$$

where $erf(x) = \frac{2}{\sqrt{\pi}}\int_0^x \exp(-x^2)dx$. Values of ***erf(x)*** are available from standard mathematical tables.

Principle of statistical (random) models:

Many spectral bands have random line positions. To approximate this type of bands, various statistical models have been developed.

Example: H_2O 6.3 μm vibrational-rotational band and H_2O rotational band are characterized by random line positions.

Assumptions: ***n*** randomly spaced lines with the mean distance δ, so that Δν= nδ ; lines are independent and have identical shapes, probability density of strength of ***i***'th line is ***p(S$_i$)***. Different ***p(S)*** give different models, for instance, Goody and Malkmus.

Strategy: derive mean transmission by multiplying transmission of each line at particular **ν**, and also integrating over probability distributions of line positions ν_i and line strength S_i for each line.

$$T_{\bar{\nu}} = \frac{1}{(\Delta\nu)^n}\int_{\Delta\nu} d\nu_1 \ldots \int_{\Delta\nu} d\nu_n \int_0^\infty p(S_1)\exp(-uS_1 f(\nu-\nu_{0,1}))dS_1 \ldots$$

$$\ldots\int_0^\infty p(S_n)\exp(-uS_n f(\nu-\nu_{0,n}))dS_n =$$

$$= \prod_{i=1}^{n} \frac{1}{\Delta\nu}\int_{\Delta\nu} d\nu_i \int_0^\infty p(S_i)\exp(-uS_i\, f(\nu-\nu_{0,i}))dS_i$$

NOTE: Above equation uses that if lines in a band are uncorrelated, the multiplication law (see Lecture 8) works for average transmittance:

$$T_{\bar{\nu},1,2} = T_{\bar{\nu},1} T_{\bar{\nu},2}$$

Since in the above equation all integral alike, we have

$$T_{\bar{\nu}} = \{\frac{1}{(\Delta\nu)} \int_{\Delta\nu} d\nu \int_{0}^{\infty} p(S)\exp(-uSf(\nu)dS\}^{n} =$$
$$= \{1 - \frac{1}{\Delta\nu} \int_{\Delta\nu} d\nu \int_{0}^{\infty} p(S)[1 - \exp(-uSf(\nu)dS\}^{n} \qquad [10.8]$$

The **mean equivalent width** may be defined as

$$\overline{W} = \int_{0}^{\infty} p(S) \int_{\Delta\nu} [1 - \exp(-uSf(\nu)] d\nu dS \qquad [10.9]$$

Recalling that $\Delta\nu = n\delta$, Eq.[10.8] can be rewritten in terms of the **mean equivalent width** giving the mean transmission as

$$T_{\bar{\nu}} = \left(1 - \frac{1}{n}\left(\frac{\overline{W}}{\delta}\right)\right)^{n} \qquad [10.10]$$

Since $\lim_{n->\infty} (1 - \frac{x}{n})^{n} -> \exp(-x)$, we have

$$\boxed{T_{\bar{\nu}} = \exp(-\frac{\overline{W}}{\delta})} \qquad [10.11]$$

NOTE: Single line transmission is **1-W/Δν,** but for many random lines it is exponential in the mean equivalent width.

Statistical (Goody) band model:

Consider a band consisting of randomly distributed Lorentz lines.

Assuming that the probability distribution of intensities is the **Poisson distribution**

$$p(S) = \bar{S}^{-1} \exp(-S / \bar{S}) \qquad [10.12]$$

where the $\bar{S}$ in the mean intensity.

$$\bar{S} = \int_0^\infty Sp(S)dS$$

For the Lorentz profile with the mean half-width α, the spectral transmittance can be expressed as

$$\boxed{T_{\bar{\nu}} = \exp\left(-\frac{\bar{S}u}{\delta}\left(1+\frac{\bar{S}u}{\pi\alpha}\right)^{-1/2}\right)} \qquad [10.13]$$

Thus, Eq.[10.13] gives the mean spectral transmittance for the Goody random model as a function of path length, ***u***, and two parameters $\frac{\bar{S}}{\delta}$ and $\frac{\bar{S}}{\alpha\pi}$.

Malkmus model: (has a higher probability of weak lines)

assumes that the probability distribution of intensities is

$$p(S) = S^{-1} \exp(-S / \bar{S})$$

and, for a Lorentz line shape, the mean transmittance is

$$\boxed{T_{\bar{\nu}} = \exp\left(-\frac{\pi\alpha}{2\delta}\left(\left(1+\frac{4\bar{S}u}{\pi\alpha}\right)^{1/2} - 1\right)\right)} \qquad [10.14]$$

Weak line limit:

For $\dfrac{\bar{S}u}{\pi\alpha} << 1$, Eq.[10.13] gives

$$T_{\bar{\nu}} = \exp\left(-\frac{\bar{S}u}{\delta}\right) \qquad [10.15]$$

Strong line limit:

For $\dfrac{\bar{S}u}{\pi\alpha} >> 1$, Eqs.[10.13] and [10.14] give

$$T_{\bar{\nu}} = \exp\left(-\frac{\sqrt{\pi\alpha\bar{S}u}}{\delta}\right) \qquad [10.16]$$

3. Curtis-Godson Approximation for inhomogeneous path.

All discussion above was for homogeneous path because band parameters are for one pressure and temperature. In real atmosphere of varying T and P some adjustments of the band models are needed to account for **inhomogeneous path** when

$$\tau = \int_u k_\nu(p(u), T(u))du$$

Strategy: reduce the radiative transfer problem to that of homogeneous path with some sort of averaged values of u*, T* and p*, so that optical depth can be computed accurately.

One-parameter scaling approximation:

Find an equivalent path u* at fixed reference $\boldsymbol{T_r}$ and $\boldsymbol{p_r}$ that results in the band model having the correct transmission.

Match optical depth for line wings (centers saturated):

$$\sum_i \frac{u^* S_i(T)\alpha_i(p_r, T_r)}{\pi(\nu - \nu_{o,i})^2} = \int_u \sum_i \frac{u S_i(T)\alpha(p,T)}{(\nu - \nu_{o,i})^2} du$$

Re-writing the half-width, α, as

$$\alpha(P,T) = \alpha(p_r,T_r)\frac{P}{P_r}\left(\frac{T_r}{T}\right)^n$$

We have

$$u^* = \int_u \left(\frac{p}{p_r}\right)\left(\frac{T_r}{T}\right)^n \rho_a ds \qquad [10.17]$$

and thus

$$\boxed{\tau_\nu = k_\nu(p_r,T_r)u^*} \qquad [10.18]$$

Two-parameter scaling approximation (Curtis-Godson approximation):

More accurate band transmission is obtained with the two-parameter approximation.

Want to find optical depth as

$$\tau = \int_u k_\nu(p,T)du = k_\nu(p^*,T^*)u \qquad [10.19]$$

Using Lorentz profile, we have

$$k_\nu(p^*,T^*) = \sum_i \tilde{S}_i \tilde{f}_{\nu,i} = \sum_i \frac{\tilde{S}_i}{\pi}\frac{\tilde{\alpha}_i}{(\nu-\nu_{0,i})^2+\tilde{\alpha}_i^{\,2}}$$

and, thus, two-adjusted parameter $\tilde{S}$ and $\tilde{\alpha}$.

They can be introduced as

$$\tilde{\bar{S}} = \int_0^u \bar{S}(T)du/u$$

$$\tilde{\alpha} = \int_0^u \bar{S}(T)\alpha(p,T)du \Big/ \int_0^u \bar{S}(T)du$$